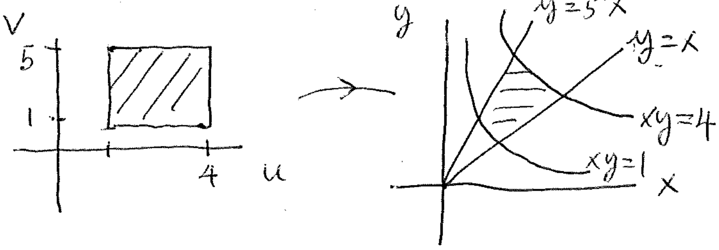
Quiz 2

Answer all questions. Each question carries 10 marks.

1. Introduce a change of variables $(u, v) \to (x, y)$ to transform the region D bounded by $xy = 1, xy = 4, y = x, y = 5x,\ x, y \geq 0$, into a rectangle in the (u, v)-plane. Indicate the boundary correspondence.

Let $u = xy$, $v = y/x$. The map $(u,v) \mapsto (x,y)$ sends the rectangle $[1,4]\times[1,5]$ to D. Their boundary correspondence =

$u=1 \Rightarrow xy=1$
$u=4 \Rightarrow xy=4$
$v=1 \Rightarrow y=x$
$v=5 \Rightarrow y=5x$



2. Find the area of the parallelogram formed by the lines $y = 2x, y = 2x+4, x+y = 1, x+y = 3$. Let $u = y - 2x$, $v = x + y$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3. \text{ So } \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}.$$

$(u,v) \mapsto (x,y)$ maps the rectangle $[0,4]\times[1,3]$ to the parallelogram.

$$\text{Its area} = \int_0^4\int_1^3 1\times\left|-\frac{1}{3}\right| dv\,du = \frac{8}{3} \ \#$$

3. Evaluate the line integral $\int_C f\, ds$ where C is the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $t \in [0, \pi]$ and $f(x, y, z) = 3z + xy$.

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$$
$$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$
$$|\vec{r}'(t)| = \left((-\sin)^2 t + \cos^2 t + 1^2\right)^{1/2} = \sqrt{2}.$$

$$\int_C f\,ds = \int_0^{\pi} (3t + \cos t \sin t)\sqrt{2}\,dt = \frac{3}{2}\sqrt{2}\,\pi^2 \ \#$$